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On the homotopy type of the spaces of spherical knots in $\mathbb{R}^n$

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Abstract. We study the spaces of embeddings $S^m \hookrightarrow \mathbb{R}^n$ and those of long embeddings $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$, i.e. embeddings of a fixed behavior outside a compact set. More precisely, we look at the homotopy fiber of the inclusion of these spaces to the spaces of immersions. We find a natural fiber sequence relating these spaces. We also compare the L_∞-algebras of diagrams that encode their rational homotopy type when the codimension $n-m$ is at least 3.

1. Introduction

In this paper, we study a relation between the following two spaces:

$$\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n) := \mathrm{hofiber}\big(\mathrm{Emb}(S^m,\mathbb{R}^n) \to \mathrm{Imm}(S^m,\mathbb{R}^n)\big), \tag{1}$$

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n) := \mathrm{hofiber}\big(\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n) \to \mathrm{Imm}_\partial(\mathbb{R}^m,\mathbb{R}^n)\big), \tag{2}$$

where $\mathrm{Emb}(-,-)$ and $\mathrm{Imm}(-,-)$ always refer to spaces of smooth embeddings and immersions, respectively. The homotopy fiber is taken over the inclusions $i_1 : S^m \subset \mathbb{R}^{m+1} \times 0^{n-m-1} \subset \mathbb{R}^n$ and $i_2 : \mathbb{R}^m = \mathbb{R}^m \times 0^{n-m} \subset \mathbb{R}^n$. The subscript ∂ means that the embeddings and immersions must coincide with the inclusion $i_2 : \mathbb{R}^m \subset \mathbb{R}^n$ outside a compact subset of $\mathbb{R}^m$. The spaces (1) and (2) are called *spaces of embeddings modulo immersions.*

The spaces $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ have been objects of active study [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 31, 32]. They were shown to be E_{m+1}-algebras [4, 7, 31] equivalent to $(m+1)$-loop spaces [3, 11, 31] when $n-m \geq 3$.

To compare their homotopy type to that of $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$, let us begin with a few observations. Given an embedding $\psi \in \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$, we can define an

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inclusion

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n) \hookrightarrow \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n). \tag{3}$$

The idea of this map is to perturb ψ near some point $p \in S^m$. By standard fibration and transversality arguments, it is easy to show that, for $n-m \geq 3$, $\pi_* \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n) \simeq \pi_* \mathrm{Emb}(S^m,\mathbb{R}^n)$ and $\pi_* \mathrm{Imm}_\partial(\mathbb{R}^m,\mathbb{R}^n) \simeq \pi_* \mathrm{Imm}(S^m,\mathbb{R}^n)$ for $* \leq 1$. This implies that the inclusion (3) induces a bijection of the sets of connected components

$$\pi_0 \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n) \simeq \pi_0 \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n).$$

It is also not hard to show that the inclusion (3) can be enhanced to an $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$-action on $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$. (To carefully define this action, the spaces $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ need to be replaced by the homotopy equivalent spaces $\overline{\mathrm{Emb}}^{\mathrm{fr}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$, $\overline{\mathrm{Emb}}^{\mathrm{fr}}(S^m,\mathbb{R}^n)$ of framed embeddings modulo framed immersions.)

Our main result now states that the homotopy quotient of $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ by the action of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ is the sphere S^{n-m-1}, or equivalently that $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ is homotopy equivalent to a principal $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$-bundle on the sphere.

Theorem 1.1. *For $n-m \geq 3$, one has an equivalence*

$$\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n) \simeq \mathrm{hofiber}\big(S^{n-m-1} \to B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)\big).$$

This in particular implies that all connected components of $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ have the same homotopy type and that $\pi_0 \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n) = \pi_0 \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$.

For an explicit definition of the classifying map $S^{n-m-1} \to B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ appearing in this theorem, we refer to Section 2.

As a consequence, we will in particular be able to express the rational homotopy types of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ through each other; see Corollaries 2.7, 2.8 below. It is furthermore well-known that the rational homotopy type of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$, $n-m \geq 3$, may be expressed through hairy graph-complexes. More precisely, in [2], hairy graph-complexes were introduced, denoted by $\mathrm{HGC}_{\bar{A}_m,n}$ in this paper, which were proved to compute the rational homotopy groups

$$H_*(\mathrm{HGC}_{\bar{A}_m,n}) \simeq \mathbb{Q} \otimes \pi_* \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n) \tag{4}$$

for $n \geq 2m+2$. The paper [12] determined the rational homotopy type of the $(m+1)$-th delooping of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$, $n-m \geq 3$. In particular, [12, Thm. 15 and Rem. 19] improved the equality (4) to the range $n-m \geq 3$. In that range, the space $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ can be disconnected, but since it is an $(m+1)$-loop space, its set of connected components forms an abelian group (in fact, finitely generated). The cited theorem proves the isomorphism (4) in degree zero as well. Note, however, that the graph-complex $\mathrm{HGC}_{\bar{A}_m,n}$ can have nontrivial homology in negative degrees, that has to be ignored. In fact, the non-positive degree homology $H_{\leq 0}(\mathrm{HGC}_{\bar{A}_m,n})$, that includes the negative degree and degree zero, is at most one-dimensional for $n-m \geq 3$.

Recently, in [13], a more general method has been developed by B. Fresse and the authors to study the rational homotopy type of (connected components of) embeddings modulo immersions spaces $\overline{\mathrm{Emb}}(L,\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}_\partial(L,\mathbb{R}^n)$, where L is either a compact submanifold of $\mathbb{R}^{m+1}$ with components of possibly different dimensions, or a closed submanifold whose unbounded connected components coincide with affine subspaces of $\mathbb{R}^{m+1}$ outside a ball of some radius R. The main result of [13] provides L_∞-algebras of diagrams that express the rational type of such spaces. (This result uses the general theory of Postnikov decompositions of (modules over) reduced operads, i.e. operads whose arity zero component is reduced to a point, which is a work in progress by M. Mienné [30]. The theory of Postnikov decompositions of operads with the empty arity zero component appeared in Mienné's thesis [29].) In particular, for the first nontrivial case of $L = S^m$, the corresponding L_∞-algebra is a hairy graph-complex denoted by $\mathrm{HGC}_{A_m,n}$.

On the rational homotopy level, the comparison of the embedding spaces $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ pursued in this paper hence translates into a comparison of the graph-complexes $\mathrm{HGC}_{\bar{A}_m,n}$ and $\mathrm{HGC}_{A_m,n}$. We shall explain in Section 3 how the relation between the spaces of Theorem 1.1 can be seen directly (and independently) on the graph-complexes, at least rationally. In fact, this is how we initially discovered Theorem 1.1. Computations from Section 3 could be useful in further pursuing the graph-complex approach from [13] applying it to other types of manifolds.

In the last section, Section 4, we study the case of codimension $n-m \leq 2$. Propositions 4.2 and 4.4 are analogs of Theorem 1.1 in codimension one and two, respectively.

2. Spherical and long embeddings

In this section, we describe how the homotopy type of $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$ is compared to that of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$, and in particular prove Theorem 1.1. Throughout Sections 2 and 3, we assume $n-m \geq 3$.

2.1. **Proof of Theorem 1.1.** The second statement of the theorem holds because the sphere S^{n-m-1} is simply connected. By the Smale–Hirsch theorem [20, 33], $\mathrm{Imm}_\partial(\mathbb{R}^m,\mathbb{R}^n) \simeq \Omega^m V_m(\mathbb{R}^n)$, where $V_m(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(n-m)$ is the Stiefel manifold of orthogonal m-frames in $\mathbb{R}^n$. Thus

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n) \simeq \mathrm{hofiber}\big(\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n) \xrightarrow{D} \Omega^m V_m(\mathbb{R}^n)\big). \tag{5}$$

The space $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ is an $(m+1)$-loop space [3, 11, 31]. We denote by $B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ its classifying space and by g the map

$$g : \Omega^m_* V_m(\mathbb{R}^n) \simeq B\Omega^{m+1} V_m(\mathbb{R}^n) \to B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$$

(where Ω_* stands for the loop space component of the constant map) obtained by applying the classifying space functor B to the inclusion $\Omega^{m+1}V_m(\mathbb{R}^n) \to \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$.

Consider also the map

$$h : S^{n-m-1} \to \Omega^m V_m(\mathbb{R}^n) \tag{6}$$

adjoint to the composition

$$\Sigma^m S^{n-m-1} = S^{n-1} \xrightarrow{h_0} \mathrm{SO}(n) \longrightarrow \mathrm{SO}(n)/\mathrm{SO}(n-m) = V_m(\mathbb{R}^n), \tag{7}$$

where h_0 is the transition map for the tangent bundle of $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$ relating trivializations over the upper and lower discs D^n_+ and D^n_-. Note that, since we assume $n-m \geq 3$, the sphere S^{n-m-1} is connected and $h(S^{n-m-1}) \subset \Omega^m_* V_m(\mathbb{R}^n)$.

To show Theorem 1.1, we will check explicitly that, for $n-m \geq 3$, one has an equivalence

$$\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) \simeq \mathrm{hofiber}\big(S^{n-m-1} \xrightarrow{g \circ h} B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)\big). \tag{8}$$

In other words, $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$ is equivalent to a principal $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$-bundle over S^{n-m-1} with the structure subgroup

$$\Omega^{m+1} V_m(\mathbb{R}^n) \subset \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n).$$

The equivalence (8) and hence Theorem 1.1 can be shown using the following two propositions.

Proposition 2.2. *For $n-m \geq 3$, one has an equivalence*

$$\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) \simeq \mathrm{hofiber}\big(\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1} \xrightarrow{m \circ (D \times h)} \Omega^m V_m(\mathbb{R}^n)\big),$$

where $m : \Omega^m V_m(\mathbb{R}^n) \times \Omega^m V_m(\mathbb{R}^n) \to \Omega^m V_m(\mathbb{R}^n)$ is a loop product.

To recall, D and h denote the maps from (5) and (6). Proposition 2.2 is related to and in fact is a consequence of Budney–Cohen's [8, Prop. 4.4]. For completeness of exposition, we present its full proof below.

Proposition 2.3. *Let $Y \xrightarrow{f} X$ be a map of pointed spaces, and let $Z \xrightarrow{h} \Omega X$ be any map. Let also $\Omega X \xrightarrow{i} \mathrm{hofiber}(Y \xrightarrow{f} X)$ denote the natural inclusion and $m : \Omega X \times \Omega X \to \Omega X$ the loop product. One has an equivalence*

$$\mathrm{hofiber}\big(Z \xrightarrow{i \circ h} \mathrm{hofiber}(Y \xrightarrow{f} X)\big) \simeq \mathrm{hofiber}(\Omega Y \times Z \xrightarrow{m \circ (\Omega f \times h)} \Omega X). \tag{9}$$

Proof of Theorem 1.1. We apply Proposition 2.2 and Proposition 2.3 to the case $Y \xrightarrow{f} X$ being $B\,\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{BD} \Omega^{m-1}_* V_m(\mathbb{R}^n)$, and $Z \xrightarrow{h} \Omega X$ being $S^{n-m-1} \xrightarrow{h} \Omega^m V_m(\mathbb{R}^n)$. One has $\Omega B\,\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ because $\pi_0\,\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$, $n-m \geq 3$, is a group [17, 18]. (Explicit deloopings of $\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$, $n-m \geq 3$, were obtained in [3, 11, 31].) Note that

$$\begin{aligned} &\mathrm{hofiber}\big(B\,\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{BD} \Omega^{m-1}_* V_m(\mathbb{R}^n)\big) \\ &\qquad \simeq \Omega^m V_m(\mathbb{R}^n) /\!\!/ \mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n). \end{aligned} \tag{10}$$

The sphere S^{n-m-1} is connected and each connected component of (10) is equivalent to $B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$, which immediately yields (8). $\square$

Proof of Proposition 2.2. Denote by $\mathrm{Emb}_*(S^m,\mathbb{R}^n)$ and $\mathrm{Imm}_*(S^m,\mathbb{R}^n)$ the spaces of embeddings and immersions, respectively, with a fixed behavior near the basepoint $*\in S^m$. One can easily see that the space

$$\overline{\mathrm{Emb}}_*(S^m,\mathbb{R}^n) := \mathrm{hofiber}\big(\mathrm{Emb}_*(S^m,\mathbb{R}^n) \overset{I}{\hookrightarrow} \mathrm{Imm}_*(S^m,\mathbb{R}^n)\big)$$

is weakly equivalent to $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$. Moreover, we claim $\mathrm{Emb}_*(S^m,\mathbb{R}^n)\simeq \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)\times S^{n-m-1}$ and $\mathrm{Imm}_*(S^m,\mathbb{R}^n)\simeq \Omega^m V_m(\mathbb{R}^n)$ with the map I of the homotopy type of $m\circ(D\times h)$.

We decompose $S^m = D^m_+ \cup_{S^{m-1}} D^m_-$, where D^m_- is a small closed disc neighborhood of the basepoint $*\in S^m$, and D^m_+ is its complementary disc. We identify $\mathbb{R}^n = S^n\setminus\{N\}$ as a sphere without its north pole. Similarly, we decompose $S^n\setminus\{N\} = (D^n_+\setminus\{N\})\cup_{S^{n-1}} D^n_-$. One has

$$\begin{aligned} &\mathrm{Emb}_*(S^m,\mathbb{R}^n)\cong \mathrm{Emb}_\partial(D^m_+, D^n_+\setminus\{N\}),\\ &\mathrm{Imm}_*(S^m,\mathbb{R}^n)\cong \mathrm{Imm}_\partial(D^m_+, S^n\setminus\{N\})\simeq \Omega^m V_m(\mathbb{R}^n). \end{aligned} \tag{11}$$

The last equivalence in (11) is by the Smale–Hirsch theorem, as the target manifold $S^n\setminus\{N\}=\mathbb{R}^n$ is contractible.

Remark 2.4. The transition map between the coordinate framing on $\mathbb{R}^n = S^n\setminus\{N\}$ and the local coordinates framing near N, when restricted on a small $(n-1)$-sphere around N, is given by the map h_0 from equation (7).

Consider the space $\mathrm{Emb}_\partial(\mathbb{R}^m\sqcup\{*\},\mathbb{R}^n)$, where $\mathbb{R}^m\sqcup\{*\}$ is given the disjoint union topology. Below, we define maps

$$\begin{tikzcd} \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)\times S^{n-m-1} \arrow[d, "A"] \arrow[dr, "B"] & \\ \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n\setminus\{0\}) \arrow[r, "C"] & \mathrm{Emb}_\partial(\mathbb{R}^m\sqcup\{*\},\mathbb{R}^n). \end{tikzcd} \tag{12}$$

The map C is the inclusion sending $f\mapsto \tilde f$, where

$$\tilde f(x) = \begin{cases} f(x), & x\in\mathbb{R}^m,\\ 0, & x = *. \end{cases}$$

By S^{n-m-1}, we understand the unit sphere in $\mathbb{R}^{n-m}$. Map B sends a pair (f,v) to $\tilde f$ such that $\tilde f(*) = 0^m\times v$ and $\tilde f|_{\mathbb{R}^m}$ is supported in the unit ball with center $-3\times 0^{m-1}$ and sending this ball inside the unit ball centered at $-3\times 0^{n-1}$. We use here the homeomorphism $\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)\cong \mathrm{Emb}_\partial(D^m,D^n)$ induced by a diffeomorphism between $\mathbb{R}^n$ and the interior $\mathrm{int}(D^n)$ of D^n, that sends $\mathbb{R}^m$ to $\mathrm{int}(D^m)$.

Finally, we define A. Let $\rho:\mathbb{R}^m\to[0,1]$ be a smooth bump function supported in the unit disc D^m. The map A sends (f,v) to $\tilde f:\mathbb{R}^m\hookrightarrow\mathbb{R}^n$ supported in the union of two unit discs with centers $-3\times 0^{m-1}$ and 0^m. Inside the first disc, $\tilde f$ is defined in the same way as in the case of map B, while inside the second disc, $\tilde f(x) = (x, -\rho(x)v)$.

Lemma 2.5. *For $n-m\geq 3$, all three maps A, B, C in* (12) *are weak homotopy equivalences. Moreover, B is homotopic to $C\circ A$.*

Proof. Consider two fibrations

$$\pi_1 : \mathrm{Emb}_\partial(\mathbb{R}^m \sqcup \{*\}, \mathbb{R}^n) \to \mathbb{R}^n,$$
$$\pi_2 : \mathrm{Emb}_\partial(\mathbb{R}^m \sqcup \{*\}, \mathbb{R}^n) \to \mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$$

obtained by restricting embeddings to one of the components $\{*\}$ or $\mathbb{R}^m$.

Since the target of π_1 is contractible, the inclusion of the fiber in the total space is an equivalence, implying that C is a weak equivalence.

The map B is a morphism of fiber bundles over $\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)$. By applying the Alexander duality and the fact that both S^{n-m-1} and the fiber of π_2 (the complement of a knot) are simply connected, we get that B induces an equivalence of fibers, therefore is an equivalence of total spaces.

It is obvious that $B\simeq C\circ A$. By the two out of three property, A is also a weak equivalence. ☐

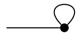
To finish the proof of Proposition 2.2, one has to show that the composition

$$\begin{aligned} J : \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)\times S^{n-m-1} &\xrightarrow[A]{\simeq} \mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n\setminus\{0\}) \\ &\xrightarrow{\cong} \mathrm{Emb}_\partial(D^m_+, D^n_+\setminus\{N\}) \\ &\longrightarrow \mathrm{Imm}_\partial(D^m_+, S^n\setminus\{N\}) \\ &\xrightarrow{\simeq} \Omega^m V_m(\mathbb{R}^n) \end{aligned}$$

is homotopic to $m\circ(D\times h)$. It is obvious that J restricted to the first factor $\mathrm{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ is homotopic to D. It follows from Remark 2.4, that J restricted on the second factor S^{n-m-1} is homotopic to h. Also, by construction, J is a concatenation of the loop obtained from the first factor with the loop obtained from the second factor, which is exactly what the formula $m\circ(D\times h)$ means. ☐

Proof of Proposition 2.3. Recall the standard construction of the homotopy fiber of a map $Y\xrightarrow{f} X$. It is the space of pairs (y,x), where $y\in Y$, $x:[0,1]\to X$ such that $x(0)=*$ and $x(1)=f(y)$. When this construction is applied, both spaces in (9) are homeomorphic to the space of triples (z,y,x), where $z\in Z$, $y\in\Omega Y$, $x:D^2\to X$ such that $x|_{\partial D^2}$ is the loop $m(h(z),(\Omega f)(y))$. ☐

2.6. **Corollaries for the rational homotopy types.** The rational homotopy $\pi_*^{\mathbb{Q}}S^{n-m-1}$ is spanned by the spherical class $\iota\in\pi^{\mathbb{Q}}_{n-m-1}S^{n-m-1}$ and the Hopf class $[\iota,\iota]\in\pi^{\mathbb{Q}}_{2n-2m-3}S^{n-m-1}$, which is nonzero only if $n-m$ is odd. The induced map in the rational homotopy $h_*:\pi_*^{\mathbb{Q}}S^{n-m-1}\to\pi_*^{\mathbb{Q}}\Omega^m V_m(\mathbb{R}^n)$ sends the spherical class ι to the $\mathrm{SO}(n)$ Euler class if n is even, and sends it to zero if n is odd. The Hopf class $[\iota,\iota]$ of S^{n-m-1} is sent to zero because the rational homotopy of any loop space is an abelian Lie algebra. Recall also that the induced map $g_* : \pi_*^{\mathbb{Q}} \Omega^m V_m(\mathbb{R}^n) \to \pi_*^{\mathbb{Q}} B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ sends the $\mathrm{SO}(n)$ Euler class to the graph-cycle

$$D = \omega$$ 

in $\mathrm{HGC}_{\bar{A}_m,n}$ (see [2, 12, 22]), which is nonzero only if n is even.

Together with Theorem 1.1, the computations above immediately imply the following corollary.

Corollary 2.7. *For $n - m \geq 3$, one has*

$$\operatorname{rk} \pi_i^{\mathbb{Q}}\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) = \operatorname{rk} \pi_i^{\mathbb{Q}}\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n),$$

except

- *for n even,* $\operatorname{rk} \pi_{n-m-2}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) = \operatorname{rk} \pi_{n-m-2}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) - 1,$
- *for n odd,* $\operatorname{rk} \pi_{n-m-1}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) = \operatorname{rk} \pi_{n-m-1}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) + 1,$
- *for $n - m$ odd,* $\operatorname{rk} \pi_{2n-2m-3}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) = \operatorname{rk} \pi_{2n-2m-3}^{\mathbb{Q}}\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) + 1.$

(It follows from Theorem 1.1 that $\pi_1\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$ is a quotient group of $\pi_1\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ and therefore is abelian.)

Any map from a suspension to an H-space is rationally coformal (and also formal). For n odd, the induced map in rational homotopy $(g \circ h)_*$ is zero, and for n even, it is nonzero only on the spherical class ι. This immediately determines the rational homotopy type of $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$, $n - m \geq 3$.

Corollary 2.8. *For $n - m \geq 3$,*

- *if $n - m$ or n is even, each component of $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$ is rationally equivalent to a product of $K(\mathbb{Q}, j)$'s; in other words, it is coformal with an abelian Quillen model;*
- *if n is odd,* $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) \simeq_{\mathbb{Q}} \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1}$.

Only in the case n odd and m even, the space $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$ is not rationally abelian. However, the failure of being nonabelian is only in the rational factor S^{n-m-1}.

3. Comparing graph-complexes

As described in the introduction, the rational homotopy types of both spaces $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ and $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$, $n - m \geq 3$, have known expressions through graph-complexes. The purpose of this section is to illustrate how Theorem 1.1 and in particular Corollaries 2.7 and 2.8 manifest themselves combinatorially on the graph-complex level. We shall proceed without using Theorem 1.1 directly, but rather by providing independent arguments, thus essentially reproving (parts of) the theorem rationally.

We will use the notion of (complete) L_∞-algebras and their Maurer–Cartan spaces. We adopt Whitehead's grading conventions in which the bracket, higher brackets, and differential of an L_∞-algebra are all of degree -1. We refer the reader to [10, Sec. 2] for a comprehensive but careful recollection, using the same grading conventions.

3.1. **Hairy graph-complexes.** In this subsection, we describe graph-complexes $\mathrm{HGC}_{\bar{A}_m,n}$, $\mathrm{HGC}_{A_m,n}$ and their L_∞-algebra structures that express the rational homotopy type, respectively, of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)$, $n-m\geq 3$. Here $\bar{A}_m$ denotes the reduced cohomology algebra $\tilde{H}^*(S^m,\mathbb{Q})$, and A_m denotes the cohomology algebra $H^*(S^m,\mathbb{Q})$. The former is spanned by a single element ω of degree m, while the latter is spanned by 1 and ω. With Whitehead's grading conventions,

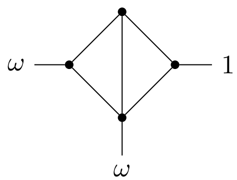
$$H_*(\mathrm{HGC}_{\bar{A}_m,n}) = \mathbb{Q}\otimes\pi_*\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n),$$
$$H_*(\mathrm{HGC}_{A_m,n}) = \mathbb{Q}\otimes\pi_*\,\overline{\mathrm{Emb}}(S^m,\mathbb{R}^n),$$

and the bracket in graph-complexes corresponds to the Whitehead bracket in the rational homotopy. Note that the latter one is almost always zero according to Corollary 2.8.

The graph-complexes are spanned by finite connected graphs with two types of vertices: external ones of valence one (called *hairs*) and internal ones of valence at least 3. Every external vertex is labeled by ω in case of $\mathrm{HGC}_{\bar{A}_m,n}$, and either by ω or by 1 in case of $\mathrm{HGC}_{A_m,n}$. Double edges and tadpoles (edges connecting a vertex to itself) are allowed. Such graphs are required to have at least one hair. Let E, V, H denote, respectively, the sets of edges, internal vertices, and ω-hairs of a graph Γ. The degree of such graph is

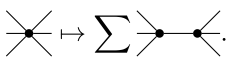
$$(n-1)\#E - n\#V - m\#H.$$

For example, the degree of the graph

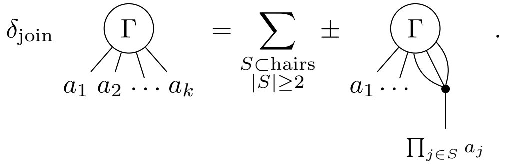


is $4n-2m-8$. Note that the edges at the hairs we also count as edges so that the diagram above has 8 edges. By an *orientation* of Γ, we understand an orientation of its edges and a linear order of its *orientation set* $E\cup V\cup H$. Changing orientation of an edge gives the sign $(-1)^n$. Changing the order of the orientation set brings in the Koszul sign of permutation, where edges are assigned degree $n-1$, internal vertices are assigned degree $-n$, and ω-hairs are assigned degree $-m$.

The differential on $\mathrm{HGC}_{\bar{A}_m,n}$ is denoted by δ_{split}; it acts by splitting the vertices into two:

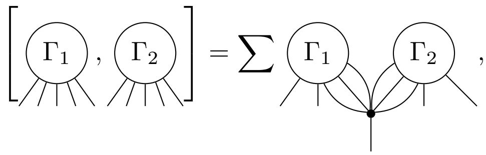
(13) $$\delta_{\mathrm{split}}\Gamma = \sum_{v\text{ vertex}} \pm\Gamma\text{ split }v$$ .

The differential on $\mathrm{HGC}_{A_m,n}$ is $\delta=\delta_{\mathrm{split}}+\delta_{\mathrm{join}}$, where δ_{split} is defined by (13), while δ_{join} joins a subset of at least two hairs into one hair, multiplying the decorations, schematically:

(14) 

Clearly, a summand in (14) is nonzero only if S contains at most one ω-hair. For the signs, note that each graph Γ' in the sums $\delta_{\mathrm{split}}\Gamma$ and $\delta_{\mathrm{join}}\Gamma$ has exactly one more vertex and one more edge than the initial graph Γ. So, to obtain an (ordered) orientation set of Γ', we just add to that of Γ the new vertex and new edge as the first and second elements. The new edge of Γ' is oriented towards its new vertex. In case of δ_{split}, there are two choices which vertex is considered as a new one, but the two resulting orientations are equivalent. With this convention, all the signs in (13) and (14) are positive.

The r-th L_∞-operation $\ell_r(\Gamma_1, \ldots, \Gamma_r)$, $r \geq 2$, is zero for $\mathrm{HGC}_{\bar{A}_m,n}$ and is defined similarly to δ_{join} for $\mathrm{HGC}_{A_m,n}$. For example, the (homotopy) Lie bracket has the following form:



where the decorations ω and 1 on hairs are multiplied whenever hairs are joined. The sum is taken over pairs of nonempty subsets of hairs of Γ_1 and Γ_2. More generally, ℓ_r is the sum over r-tuples of nonempty subsets of hairs of $\Gamma_1, \ldots, \Gamma_r$ with every summand being a new connected hairy graph, where all selected hairs are joined into one. With our grading conventions, each operation ℓ_r (as well as the differential) has degree -1. The orientation of each graph in the sum is obtained by concatenating the orientation sets of $\Gamma_1, \ldots, \Gamma_r$, and placing the new vertex and new edge in front. The new (hair) edge is again oriented upward—towards the new vertex.

3.2. **Connected components and Maurer–Cartan elements.** As it is explained in the introduction, and also stated in Theorem 1.1, one has that

$$\pi_0\,\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) = \pi_0\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n), \quad n - m \geq 3, \tag{15}$$

are isomorphic as (abelian) groups, and all components of $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$ (as well as all components of $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$) have the same homotopy type. These groups are almost always finite except for two cases:
(a) $m = 2k - 1$, $n = 4k - 1$, $k \geq 2$,
(b) $m = 4k - 1$, $n = 6k$, $k \geq 1$.
In these two cases, this group is infinite of rank one [12, Cor. 20]. This fact can also be easily obtained from Haefliger's [18, Cor. 6.7, Rem. 6.8]. In case (a),

an infinite order generator appears as image, under inclusion

$$\Omega^{2k} V_{2k-1}(\mathbb{R}^{4k-1}) \to \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{2k-1}, \mathbb{R}^{4k-1}),$$

of the $\mathrm{SO}(2k)$ Euler class in

$$\pi_{2k} V_{2k-1}(\mathbb{R}^{4k-1}) = \pi_{2k}\big(\mathrm{SO}(4k-1)/\mathrm{SO}(2k)\big).$$

In case (b), an infinite order generator corresponds to the Haefliger trefoil $S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ (see [17, 18]).

By [13, Cor. 1.3], the L_∞-algebras $\mathrm{HGC}_{\bar{A}_m,n}$ and $\mathrm{HGC}_{A_m,n}$ do provide some information about the sets (15). Namely, one has naturally defined finite-to-one maps

$$m : \pi_0\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \to \mathsf{MC}(\mathrm{HGC}_{\bar{A}_m,n})/\!\sim,$$
$$m : \pi_0\, \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n) \to \mathsf{MC}(\mathrm{HGC}_{A_m,n})/\!\sim$$

from the sets of connected components to the sets of Maurer–Cartan elements modulo gauge equivalence. Here "finite" can also mean zero, i.e. some components are not hit. Since the L_∞-algebra $\mathrm{HGC}_{\bar{A}_m,n}$ is abelian,

$$\mathsf{MC}(\mathrm{HGC}_{\bar{A}_m,n})/\!\sim\; = H_0(\mathrm{HGC}_{\bar{A}_m,n}).$$

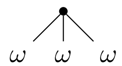
It is not hard to see that $\mathrm{HGC}_{A_m,n}$ in degrees at most 0 can have only trees with all hairs labeled by ω [13, Prop. 5.1]. Thus, $H_0(\mathrm{HGC}_{A_m,n}) = H_0(\mathrm{HGC}_{\bar{A}_m,n})$ and $\mathsf{MC}(\mathrm{HGC}_{A_m,n})/\!\sim\; = \mathsf{MC}(\mathrm{HGC}_{\bar{A}_m,n})/\!\sim$; see [13, Cor. 5.2]. By [12, Rem. 19],

$$\mathbb{Q} \otimes \pi_0\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \simeq H_0(\mathrm{HGC}_{\bar{A}_m,n}).$$

The latter group is nontrivial (and is $\mathbb{Q}$) exactly in the two cases (a) and (b) above. The Maurer–Cartan elements corresponding to case (a) are multiples of the line graph

$$L_\omega = \ \omega \text{ --- } \omega\ .$$

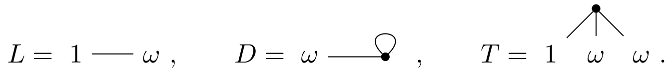
For case (b), such elements are multiples of the tripod

$T_\omega =$  .

By [13, Cor. 1.3], for an embedding $\psi \in \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ (respectively, $\psi \in \overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$), the rational homotopy type of the component $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)_\psi$ (respectively, $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)_\psi$) is expressed by the positive degree truncation of the $m(\psi)$ twisted L_∞-algebra $(\mathrm{HGC}^{m(\psi)}_{\bar{A}_m,n})_{>0}$ (respectively, $(\mathrm{HGC}^{m(\psi)}_{A_m,n})_{>0}$); see [10, Sec. 2.1] for the definition of the twisted L_∞-structure on an L_∞-algebra. Since the L_∞-algebra $\mathrm{HGC}_{\bar{A}_m,n}$ is abelian, such a twist has no effect on it, which corresponds to the fact that all connected components of a loop space have the same homotopy type. In case (a), the twist by L_ω changes neither the differential nor the bracket of $\mathrm{HGC}_{A_{2k-1},4k-1}$. This is because, for even codimension $n-m$, any graph with two ω-hairs attached to an internal vertex is zero. The twisting by T_ω does affect the differential and the L_∞ structure of $\mathrm{HGC}_{A_{4k-1},6k}$. We do not do it here, but one can show that $\mathrm{HGC}^{T_\omega}_{A_{4k-1},6k}$ is L_∞ isomorphic to the non-deformed one $\mathrm{HGC}_{A_{4k-1},6k}$, which confirms the fact that all components of $\overline{\mathrm{Emb}}(S^m, \mathbb{R}^n)$, $n-m \geq 3$, have the same homotopy type.

3.3. **Computations.** The relation between the long and non-long embedding spaces can be reproduced combinatorially on graph-complexes as follows. There is an obvious inclusion of L_∞-algebras $\mathrm{HGC}_{\bar{A}_m,n} \to \mathrm{HGC}_{A_m,n}$ corresponding to the inclusion $\bar{A}_m \to A_m$.

Let us also consider the following low degree diagrams which are nonzero for certain values of m and n:



The graph L is of degree $n-m-1$ and is always nonzero; D is of degree $n-m-2$ and is nonzero if n is even; and T is of degree $2n-2m-3$ and is nonzero if and only if $n-m$ is odd. One has that $dL = D$, i.e., for even n, the corresponding classes cancel in homology in $\mathrm{HGC}_{A_m,n}$ (but not in $\mathrm{HGC}_{\bar{A}_m,n}$ since $L \notin \mathrm{HGC}_{\bar{A}_m,n}$). Also note that $dT = 0$. Using these classes, we can completely describe the relation between $\mathrm{HGC}_{A_m,n}$ and $\mathrm{HGC}_{\bar{A}_m,n}$ as follows.

Theorem 3.4. *The mapping cone C of the inclusion $\mathrm{HGC}_{\bar{A}_m,n} \to \mathrm{HGC}_{A_m,n}$ has the following homology, depending on m and n.*

- *For m,n even, $H(C)$ is one-dimensional, spanned by a class whose projection to $\mathrm{HGC}_{\bar{A}_m,n}$ is D.*
- *For n even and m odd, $H(C)$ is two-dimensional, spanned by a class corresponding to D in $\mathrm{HGC}_{\bar{A}_m,n}$ as before and the class of $T \in \mathrm{HGC}_{A_m,n}$.*
- *For m,n odd, $H(C)$ is one-dimensional, spanned by the class of L in $\mathrm{HGC}_{A_m,n}$.*
- *For n odd and m even, $H(C)$ is two-dimensional, spanned by the class of L and T in $\mathrm{HGC}_{A_m,n}$.*

Remark 3.5. Theorem 3.4 provides a different proof of Corollary 2.7.

The result can alternatively be reformulated as follows.

Corollary 3.6. *Let $U^t \subset \mathrm{HGC}_{A_m,n}$ be the subspace spanned by trees with exactly one 1-decorated hair. Consider the vector space direct sum*

$$U^t \oplus \mathrm{HGC}_{\bar{A}_m,n} \subset \mathrm{HGC}_{A_m,n} \tag{16}$$

with the induced (subspace) L_∞-structure. Then the inclusion (16) *is a quasi-isomorphism of L_∞-algebras.*

As an immediate consequence, the L_∞-algebra $\mathrm{HGC}_{A_m,n}$ is homotopy abelian for $n-m$ even. Indeed, for $n-m$ even, there can be at most one ω-hair attached to a vertex by symmetry. (In particular, this means that U^t is one-dimensional and is spanned by L.) But then the statement easily follows from Corollary 3.6 since all possible higher L_∞-operations necessarily produce multiple ω-hairs at some vertex. Less trivially, the above arguments can also be extended to show that $\mathrm{HGC}_{A_m,n}$ is homotopy abelian for n even and m odd. This gives a different proof of the first statement of Corollary 2.8. Similarly, we can also recover the second statement of Corollary 2.8, which is immediate in case both m and n are odd. In the remaining case n odd and m even, there is a nontrivial bracket, namely $[L, L] = T$, so that $\mathrm{HGC}_{A_m,n}$ is not homotopy abelian. It is possible to upgrade the map Φ that we construct below (see Lemma 3.9) to an L_∞-map (see the footnote at the end of the proof of Lemma 3.9) that would allow one to split off L and T—the two classes coming from S^{n-m-1}—as an L_∞-direct summand.

To prepare for the proof of Theorem 3.4, let us introduce the nonunital dgca

$$A'_m = \mathbb{Q}\epsilon \oplus \mathbb{Q}\omega$$

with ϵ of degree 0 and ω of degree m, and products $\epsilon^2 = \epsilon$ and $\epsilon\omega = \omega^2 = 0$. We consider the hairy graph-complex $\mathrm{HGC}_{A'_m,n}$. Note also that the complexes $\mathrm{HGC}_{A'_m,n}$ and $\mathrm{HGC}_{A_m,n}$ are isomorphic as graded vector spaces, identifying ϵ and 1. In fact, from now on, we shall tacitly identify the decorations ϵ and 1 on hairs of graphs, keeping in mind however that the differentials on $\mathrm{HGC}_{A'_m,n}$ and $\mathrm{HGC}_{A_m,n}$ are different. Concretely, the differential in $\mathrm{HGC}_{A_m,n}$ has pieces fusing several 1-decorated hairs with one ω-decorated hair, and these terms are absent in the differential on $\mathrm{HGC}_{A'_m,n}$. Note that there is again an inclusion $\mathrm{HGC}_{\bar{A}_m,n} \to \mathrm{HGC}_{A'_m,n}$.

Lemma 3.7. *The inclusion map $\mathbb{Q}L \oplus \mathbb{Q}T \oplus \mathrm{HGC}_{\bar{A}_m,n} \to \mathrm{HGC}_{A'_m,n}$ is a quasi-isomorphism. Here we understand that $\mathbb{Q}T := 0$ in case $n - m$ is even since then $T = 0$.*

Remark 3.8. It follows from [13, Cor. 1.3] that the complex $\mathrm{HGC}_{A'_m,n}$ computes the rational homotopy groups of the space $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m \sqcup \{*\}, \mathbb{R}^n)$. On the other hand, by Lemma 2.5, $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m \sqcup \{*\}, \mathbb{R}^n) \simeq \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1}$. This explains the quasi-isomorphism of Lemma 3.7.

Proof of Lemma 3.7. There is a splitting of complexes

$$\mathrm{HGC}_{A'_m,n} = \mathrm{HGC}_{\bar{A}_m,n} \oplus U$$

with U being the subcomplex spanned by graphs with at least one hair labeled ϵ. Our goal is to show that $H(U)$ is one- or two-dimensional. To do this, we may follow the proof of [23, Thm. 1]. First note that the differential creates exactly one internal vertex; hence the homology of U is graded by the number of internal vertices. Let $U^t \subset U$ be the subcomplex spanned by trees with exactly one hair labeled ϵ. It is an easy exercise to check that $H(U^t) = \mathbb{Q}L \oplus \mathbb{Q}T$. We are going to show by induction on the number of internal vertices that the inclusion $U^t \subset U$ is a quasi-isomorphism.

For zero internal vertices, the statement is obvious—the part of the cohomology with zero internal vertices is spanned by L on either side. Suppose we know the statement for less than k internal vertices, and we desire to prove it for k internal vertices. Consider the splitting $U = U_1 \oplus U_{>1}$, where U_1 is spanned by diagrams having exactly one ϵ-labeled hair, and $U_{>1}$ being spanned by diagrams having at least two such hairs. The space U_1 is preserved by the differential. One may set-up a bounded spectral sequence such that the lowest page differential is the component $f : U_{>1} \to U_1$ that creates one new internal vertex with an ϵ-hair, connecting all ϵ-hairs to it. Indeed, the complex

$\mathrm{HGC}_{A'_m,n}$ as well as U is a direct sum of finite complexes as the differential preserves the number of edges minus the number of internal vertices. This number is sometimes called *complexity*, and the number of graphs in $\mathrm{HGC}_{A'_m,n}$ of any given complexity is finite. To obtain the spectral sequence in question, we filter U (or rather all its complexity summands) by ρ minus the number of vertices, where we set $\rho(U_1) = 1$ and $\rho(U_{>1}) = 0$. The differential d_0 of the induced spectral sequence is exactly the map f.

The map f is injective. The cokernel $V := \operatorname{coker} f$ consists of L and graphs which become disconnected upon removing the vertex at the ϵ-hair.

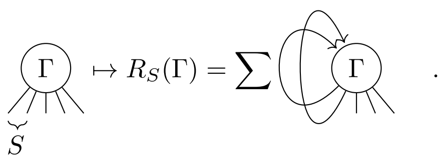
Going further, one may filter V by the number of connected components at that vertex. On the associated graded, the complex obtained is just a symmetric power of the complex U.

Parallelly, we may restrict the filtrations above to the subcomplex $U^t \subset U$, and also consider the associated spectral sequences. The inclusion $U^t \subset U$ furthermore induces a map between the spectral sequences for U^t and U. By standard spectral sequence comparison results, we have successfully completed the induction step if the induced map of spectral sequences is a quasi-isomorphism on some page, at least up to k internal vertices.

But on the final page considered above, this induced map is identified with the symmetric product of the inclusion $U^t \to U$, and by the induction hypothesis, the symmetric power of the inclusion induces an isomorphism on the part of the cohomology with at most k internal vertices so that we are done. $\square$

Our next goal is to compare the complexes $\mathrm{HGC}_{A'_m,n}$ and $\mathrm{HGC}_{A_m,n}$. Note that the algebra A'_m cannot be obtained as an associated graded of A_m. Neither $\mathrm{HGC}_{A'_m,n}$ is an associated graded of $\mathrm{HGC}_{A_m,n}$. So we need a more subtle argument to compare their homology. Our strategy will be to split each of the two complexes into three pieces: an acyclic one, a small piece where the two complexes differ, and the *main part* that we show to be the same up to a nontrivial isomorphism. To this end, we will consider the subcomplexes $\mathrm{HGC}'_{A'_m,n} \subset \mathrm{HGC}_{A'_m,n}$ and $\mathrm{HGC}'_{A_m,n} \subset \mathrm{HGC}_{A_m,n}$ spanned by all the diagrams with at least one ω-labeled hair excluding L and D. These subcomplexes are our *main parts*.

Consider now a hairy graph $\Gamma \in \mathrm{HGC}'_{A'_m,n}$, and let S be some subset of the hairs decorated by ϵ in Γ. Denote by $R_S(\Gamma)$ the sum of all graphs obtained by reconnecting the hairs in S to internal vertices of Γ, not forming tadpoles (i.e., a hair cannot be connected to the internal vertex it attaches to), pictorially,



Note that each graph in the sum has the same orientation set (of vertices, edges, and ω-labels). So we keep the same order of their orientation sets and the same orientation of edges. With this convention, no signs appear in the sum above.

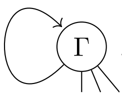
Now consider the map $\Phi : \mathrm{HGC}'_{A'_m,n} \to \mathrm{HGC}'_{A_m,n}$ which is defined combinatorially by the formula

$$\Phi(\Gamma) = (-1)^{\#\epsilon} \sum_S R_S(\Gamma),$$

where $\Gamma \in \mathrm{HGC}'_{A'_m,n}$ is a graph with $\#\epsilon$ many ϵ-decorated hairs.

Lemma 3.9. *The map $\Phi : \mathrm{HGC}'_{A'_m,n} \to \mathrm{HGC}'_{A_m,n}$ is an isomorphism of complexes.*

Proof. It is clear that the map is an isomorphism of graded vector spaces since $\Phi(\Gamma) = \pm\Gamma + (\cdots)$, with $(\cdots)$ representing terms of loop orders higher than that of Γ. We next show that Φ commutes with the differentials. The only graph in $\mathrm{HGC}'_{A'_m,n}$ that does not have internal vertices is $L_\omega = \omega$ —— ω . Since $d'(L_\omega) = d(L_\omega) = 0$ and $\Phi(L_\omega) = L_\omega$, this graph can be ignored, and from now on, we only consider graphs that have internal vertices. Let us first reformulate the problem. We identify $\mathrm{HGC}'_{A'_m,n}$ and $\mathrm{HGC}'_{A_m,n}$ as graded vector spaces, and denote the differential of $\mathrm{HGC}'_{A'_m,n}$ by d' and that of $\mathrm{HGC}'_{A_m,n}$ by d. Let $s : \mathrm{HGC}'_{A'_m,n} \to \mathrm{HGC}'_{A'_m,n}$ be the map of graded vector spaces that reconnects one hair h labeled ϵ to an internal vertex (but not the one from which h is growing):

$s(\Gamma) = \sum$  .

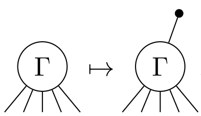
Then we can write $\Phi = \exp(s) \circ I_\epsilon$, where $I_\epsilon(\Gamma) = (-1)^{\#\epsilon}\Gamma$. We desire to show that $\Phi \circ d' = d \circ \Phi$, or equivalently,

$$\exp(\mathrm{ad}_s)(\underbrace{I_\epsilon d' I_\epsilon}_{=:\bar{d}'}) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathrm{ad}_s^j \bar{d}' \stackrel{?}{=} d,$$

where $\bar{d}' = I_\epsilon d' I_\epsilon$ and $\mathrm{ad}_s = [s, -]$ is the commutator as usual.

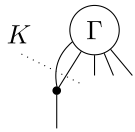
Furthermore, let us split the differential d' and similarly $\bar{d}'$ in several pieces. To this end, it is most convenient to temporarily enlarge our complex $\mathrm{HGC}'_{A'_m,n}$ in that we also allow graphs with univalent and bivalent internal vertices. Then we split $d' = d'_1 - B_\varnothing + d'_\epsilon + d'_\omega$ into the following four terms.

- d'_1 splits a vertex into two vertices, distributing the incoming edges in all possible ways, including those that create uni- or bivalent internal vertices.
- $B_\varnothing$ attaches a new univalent vertex to the graph. The sign is such that it precisely cancels those terms from d'_1 that create univalent internal vertices:

 .

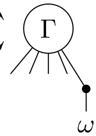
- d'_ϵ creates a new internal vertex with an ϵ-decorated hair and attaches a nonempty subset of the ϵ-decorated hairs to it:

$$d'_\epsilon\Gamma = \sum_{\substack{K \\ |K|\geq 1}} A_K(\Gamma), \qquad A_K(\Gamma) = $$



- d'_ω creates a bivalent internal vertex on an ω-decorated hair:

$$d'_\omega\Gamma = C_\varnothing(\Gamma), \qquad C_\varnothing(\Gamma) = \sum$$



Each operation above produces a sum of graphs Γ' that have one more vertex and one more edge than the graph Γ. So we put these two new elements as the first and second elements of the orientation set of Γ' keeping without change the rest. The new edge in Γ' is always oriented towards the new vertex. In the case of d'_ω (as well as in the case of d'_ϵ), the new edge is considered to be the hair one.

Note that the $|K| = 1$-terms of d'_ϵ and d'_ω together cancel all terms in the total differential d' that possibly create a graph with a bivalent internal vertex.

Finally, we note that $I_\epsilon d'_1 I_\epsilon = d'_1$, $I_\epsilon B_\varnothing I_\epsilon = B_\varnothing$, and $I_\epsilon d'_\omega I_\epsilon = d'_\omega$. Denoting $\bar{d}'_\epsilon := (I_\epsilon d'_\epsilon I_\epsilon)$, we furthermore have

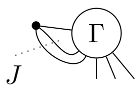
$$\bar{d}'_\epsilon\Gamma = \sum_{\substack{K \\ |K|\geq 1}} (-1)^{|K|-1} A_K(\Gamma).$$

One quickly checks that $[s, d'_1] = 0$. Note that, in $s(d'_1(\Gamma))$, the part that comes from connecting an ϵ-hair h to a new vertex created by d'_1 by blowing up the vertex to which h is attached is zero. Indeed, when n is even, each such graph is zero as it contains a double edge. When n is odd, the sum can be seen as a sum of pairs of identical graphs with an edge, former h, appearing with the opposite orientation. Thus, two such graphs cancel each other. Furthermore,

$$\frac{1}{j!}(\mathrm{ad}_s^j B_\varnothing)(\Gamma) = \sum_{\substack{J \\ |J|=j}} B_J(\Gamma), \tag{17}$$

where the sum is over subsets J of the set of ϵ-labeled hairs and

$$B_J(\Gamma) = \sum$$



is obtained by connecting the hairs J to a new vertex and that furthermore to an arbitrary existing vertex of Γ. Indeed, if we denote by $B_j(\Gamma)$ the right-hand side of (17), one has

$$s(B_j(\Gamma)) = B_j(s(\Gamma)) + (j+1)B_{j+1}(\Gamma),$$

i.e., $[s, B_j] = (j+1)B_{j+1}$, which applying induction proves (17). Next,

$$\frac{1}{j!}(\mathrm{ad}_s^j \bar{d}'_\epsilon)(\Gamma) = \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|=j, |K|\geq 1}} (-1)^{|K|-1} A_{J\cup K}(\Gamma) + \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|=j-1, |K|\geq 1}} (-1)^{|K|-1} B_{J\cup K}(\Gamma).$$

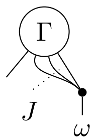
To prove it, denote by $A_{j;>0}(\Gamma)$ the first sum and by $B_{j-1;>0}(\Gamma)$ the second one. The above is proved by checking that the operations $A_{j;>0}$ and $B_{j-1;>0}$ satisfy

$$[s, A_{j;>0}] = (j+1)A_{j+1;>0} + B_{j;>0}, \quad [s, B_{j-1;>0}] = jB_{j;>0}.$$

Finally,

$$\frac{1}{j!}(\mathrm{ad}_s^j d'_\omega)(\Gamma) = \sum_{\substack{J \\ |J|=j}} C_J(\Gamma),$$

where the sum is again over subsets J of the ϵ-decorated hairs, and $C_J(\Gamma)$ is obtained by connecting the hairs in J to one new vertex attached to an ω-decorated hair:

$C_J(\Gamma) = \sum$  .

Now, putting everything together, we get (with the sums being over subsets of the ϵ-decorated hairs)

$$\begin{aligned}(\Phi d' \Phi^{-1})(\Gamma) &= \sum_{j=0}^{\infty} \frac{1}{j!}(\mathrm{ad}_s^j \bar{d}')(\Gamma) \\ &= d'_1(\Gamma) - \sum_{\substack{J \\ |J|\geq 0}} B_J(\Gamma) + \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|\geq 0, |K|\geq 1}} (-1)^{|K|-1} A_{J\cup K}(\Gamma) \\ &\qquad + \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|\geq 0, |K|\geq 1}} (-1)^{|K|-1} B_{J\cup K}(\Gamma) + \sum_{\substack{J \\ |J|\geq 0}} C_J(\Gamma) \\ &= d'_1(\Gamma) - \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|, |K|\geq 0}} (-1)^{|K|} B_{J\cup K}(\Gamma) \\ &\qquad - \sum_{\substack{J,K \\ J\cap K=\varnothing \\ |J|\geq 0, |K|\geq 1}} (-1)^{|K|} A_{J\cup K}(\Gamma) + \sum_{\substack{J \\ |J|\geq 0}} C_J(\Gamma).\end{aligned}$$

Now we use (twice) that, for any function $J \mapsto X_J$ on subsets as above,

$$\sum_{J\cap K=\varnothing} (-1)^{|K|} X_{J\cup K} = X_\varnothing.$$

This simplifies the above expression to

$$d_1'(\Gamma) - B_\varnothing(\Gamma) + \sum_{\substack{J \\ |J|\geq 1}} A_J(\Gamma) + \sum_{\substack{J \\ |J|\geq 0}} C_J(\Gamma) = d(\Gamma).$$

This is precisely d; hence the lemma is proven.[1] □

Let us finish the proof of Theorem 3.4.

Proof of Theorem 3.4. Let $\mathrm{HGC}'_{\bar{A}_m,n} \subset \mathrm{HGC}_{\bar{A}_m,n}$ be the subcomplex spanned by all the diagrams excluding D. Note that we have a natural inclusion $\mathrm{HGC}'_{\bar{A}_m,n} \to \mathrm{HGC}'_{A'_m,n}$, fitting into the commutative diagram

$$\begin{tikzcd} \mathrm{HGC}'_{\bar{A}_m,n} \arrow[r] \arrow[dr] & \mathrm{HGC}'_{A'_m,n} \arrow[d, "\Phi", "\cong"'] \\ & \mathrm{HGC}'_{A_m,n}. \end{tikzcd}$$

From Lemma 3.7 and its proof, we see that

$$H(\mathrm{HGC}'_{A_m,n}) \cong H(\mathrm{HGC}'_{A'_m,n}) \cong H(\mathrm{HGC}'_{\bar{A}_m,n}) \oplus \mathbb{Q}T,$$

again using the convention that $\mathbb{Q}T = 0$ if $T = 0$.

To show Theorem 3.4, it just remains to compare the homology of $\mathrm{HGC}'_{A_m,n}$ and $\mathrm{HGC}_{A_m,n}$. The complex $\mathrm{HGC}_{A_m,n}$ is a direct sum of three complexes

$$\mathrm{HGC}_{A_m,n} = W_0 \oplus \mathrm{HGC}'_{A_m,n} \oplus \big(\mathbb{Q}L \oplus \mathbb{Q}D\big), \tag{18}$$

with W_0 spanned by graphs with zero ω-vertices. It is shown in [23, Thm. 1] (see also Lemma 3.7) that $H(W_0) = 0$. Now the last summand in (18) has a nontrivial differential (sending L to D) if and only if n is even, i.e. if and only if $D \neq 0$. Combining the above observations, depending on the parity of n and $n - m$, we arrive at Theorem 3.4. □

4. Codimension $n - m \leq 2$

4.1. **Codimension one.**

Proposition 4.2. *For $n \geq 2$, one has an equivalence*

$$\overline{\mathrm{Emb}}(S^{n-1}, \mathbb{R}^n) \simeq \begin{cases} S^0 \times \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1}, \mathbb{R}^n), & n = 3 \text{ or } 7, \\ \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1}, \mathbb{R}^n) & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that Proposition 2.2 holds for $n - m = 1$. The crucial fact is that the complement of any long knot $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ is homotopy equivalent to S^0, which follows from the generalized Schoenflies theorem [28]. Any codimension one long knot is regularly homotopic to the trivial one [21, Thm. 2], which means that the induced map

$$D_* : \pi_0 \,\mathrm{Emb}_\partial(\mathbb{R}^{n-1}, \mathbb{R}^n) \to \pi_0 \Omega^{n-1} V_{n-1}(\mathbb{R}^n) = \pi_{n-1}\mathrm{SO}(n)$$

[1] The same construction applied to disconnected graphs, interpreted as the Chevalley complex, in fact can be used to construct an L_∞-isomorphism, not just one of complexes.

is zero. On the other hand, the map $h_*:\pi_0 S^0 \to \pi_{n-1}\mathrm{SO}(n)$ is trivial if and only if S^{n-1} can be reversed in $\mathbb{R}^n$, i.e., if and only if $n=3$ or 7 [33, 21]. The result follows. $\square$

It is known that $\pi_0\,\mathrm{Emb}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)=\Theta_n$, $n\neq 4$—the group of n-spheres; see [5, Sec. 5] and references within. (For $n=4$, the question whether the space is connected is equivalent to the smooth Schoenflies problem—does a smoothly embedded S^3 in $\mathbb{R}^4$ always bound the standard D^4—which is still open in this dimension. To see that $\pi_0\,\mathrm{Emb}_\partial(\mathbb{R}^4,\mathbb{R}^5)=0$, in addition to the argument given in [5, Sec. 5], one has to use the fact that the set of pseudoisotopy classes of relative to the boundary diffeomorphisms of D^4 is trivial by [25, Thm. 1].) By the same argument as in the proof of Theorem 1.1, one gets

$$\overline{\mathrm{Emb}}(S^{n-1},\mathbb{R}^n)\simeq \mathrm{hofiber}\big(S^0\to B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)\times\pi_{n-1}\mathrm{SO}(n)\big),\quad n\neq 4,$$

since $\Omega^{n-1}\mathrm{SO}(n)\mathbin{/\!\!/}\mathrm{Emb}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)\simeq B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)\times\pi_{n-1}\mathrm{SO}(n)$. We conclude that the main statement of Theorem 1.1 holds for $n=m+1$ if and only if $n=3$ or 7.

4.3. **Codimension two.** The main statement of Theorem 1.1 always fails in codimension $n-m=2$. There are two reasons for this. Firstly, for $n\geq 3$, neither $\mathrm{Emb}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$ nor $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$ are loop spaces [5, Prop. 5.11].[2] The problem is that most of the knots are not invertible (see [34, 35, 27]). Thus, a space with an $\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$-action is not the same as a principal $\Omega B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$-bundle, the latter notion being applicable to the homotopy fiber space of a map to $B\,\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$. Secondly, the complement C_f of a long knot $f:\mathbb{R}^{n-2}\hookrightarrow\mathbb{R}^n$ almost always is not weakly equivalent to S^1 (see [34]). Thus, Proposition 2.2 does not hold for $n=m+2$. Indeed, the space $\mathrm{Emb}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n\setminus\{0\})$ is weakly homotopy equivalent to $\mathrm{Emb}_\partial(\mathbb{R}^{n-2}\sqcup\{*\},\mathbb{R}^n)$ (by the same argument as in Lemma 2.5), but the latter space is a possibly nontrivial fiber bundle over $\mathrm{Emb}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$ with fiber C_f.

Nonetheless, if we consider only knots that have (homotopy) inverses, the analog of Theorem 1.1 holds. Let $\mathrm{Emb}_\partial^\times(\mathbb{R}^{n-2},\mathbb{R}^n)\subset\mathrm{Emb}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$ and $\mathrm{Emb}^\times(S^{n-2},\mathbb{R}^n)\subset\mathrm{Emb}(S^{n-2},\mathbb{R}^n)$ be the unions of components corresponding to invertible elements in $\pi_0\,\mathrm{Emb}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)=\pi_0\,\mathrm{Emb}(S^{n-2},\mathbb{R}^n)$. Let also $\overline{\mathrm{Emb}}{}_\partial^\times(\mathbb{R}^{n-2},\mathbb{R}^n)\subset\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)$ and $\overline{\mathrm{Emb}}{}^\times(S^{n-2},\mathbb{R}^n)\subset\overline{\mathrm{Emb}}(S^{n-2},\mathbb{R}^n)$ be their preimage spaces.

Proposition 4.4. *For $n\geq 3$, one has an equivalence*

$$\overline{\mathrm{Emb}}{}^\times(S^{n-2},\mathbb{R}^n)\simeq\mathrm{hofiber}\big(S^1\to B\,\overline{\mathrm{Emb}}{}_\partial^\times(\mathbb{R}^{n-2},\mathbb{R}^n)\big).$$

As a consequence, all components in $\overline{\mathrm{Emb}}{}^\times(S^{n-2},\mathbb{R}^n)$ have the same homotopy type.

[2]It was pointed out to us by R. Budney that the proof of this proposition has a little mistake that can easily be corrected. Contrary to what is said, there are codimension two long knots f with exterior $C_f\not\simeq S^1$ and $\pi_1 C_f=\mathbb{Z}$. Such knots are studied in [19].

Proof. The only thing that remains to be shown is that the complement C_f of any invertible long knot $f:\mathbb{R}^{n-2}\hookrightarrow\mathbb{R}^n$ is homotopy equivalent to S^1. Let $\bar{f}$ be in a component inverse to the component of f. One has $*\simeq\tilde{C}_{f\cdot\bar{f}}\simeq\tilde{C}_f\vee\tilde{C}_{\bar{f}}$, where $\tilde{C}_g$ denotes the infinite cyclic cover of C_g. Since a retract of a contractible space is contractible, $\tilde{C}_f\simeq *$ and therefore $C_f=\tilde{C}_f/\mathbb{Z}\simeq S^1$. □

In fact, for $n\neq 4$, $C_f\simeq S^1$ if and only if $f:\mathbb{R}^{n-2}\hookrightarrow\mathbb{R}^n$ is isotopic to a (re)parametrization of the trivial knot [26, Thm. (3)], [37, Cor. 3.1], [38, Thm. 16.1]. (For $n=4$, it is an open question, as it is neither known whether there are invertible knots different from the trivial one nor whether the complement being a homotopy circle implies the knot is invertible.) Moreover, $\pi_0\operatorname{Emb}^{\times}_{\partial}(\mathbb{R}^{n-2},\mathbb{R}^n)=\Theta_{n-1}$, $n\neq 4$; see [5, Prop. 5.11].

4.5. **Goodwillie–Weiss calculus and graph-complexes.** Given a (formally) immersed manifold M in $\mathbb{R}^n$, one can consider the functor $\overline{\operatorname{Emb}}(-,\mathbb{R}^n)$ and its objectwise rationalization $\overline{\operatorname{Emb}}(-,\mathbb{R}^n)^{\mathbb{Q}}$ on the poset of open sets of M. Goodwillie–Weiss calculus [16, 39] produces Taylor towers of approximations to these two functors:

$$\overline{\operatorname{Emb}}(M,\mathbb{R}^n)\to T_\infty\overline{\operatorname{Emb}}(M,\mathbb{R}^n)\to T_\infty\overline{\operatorname{Emb}}(M,\mathbb{R}^n)^{\mathbb{Q}}. \tag{19}$$

In case codimension is at least 3, the first map is an equivalence [14, 15], and the second map is finite-to-one on π_0 and a rational equivalence on connected components [12, Sec. 4.2]. Even when the codimension condition is not satisfied, it can still be interesting to know what is the right-hand side space of (19) as it can provide interesting invariants or more generally cohomology classes of the embedding space in question. In [13, Thm. 1.1], B. Fresse and the authors computed $T_\infty\overline{\operatorname{Emb}}(M,\mathbb{R}^n)^{\mathbb{Q}}$ expressing it as the simplicial set of Maurer–Cartan elements of associated L_∞-algebra of hairy graph-complexes, provided M is immersible or formally immersible in $\mathbb{R}^{n-2}$. In particular, one has

$$T_\infty\overline{\operatorname{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)^{\mathbb{Q}}\simeq\mathsf{MC}_\bullet(\mathrm{HGC}_{\bar{A}_m,n}),\quad n-m\geq 2; \tag{20}$$

$$T_\infty\overline{\operatorname{Emb}}(S^m,\mathbb{R}^n)^{\mathbb{Q}}\simeq\mathsf{MC}_\bullet(\mathrm{HGC}_{A_m,n}),\quad \begin{array}{l} n-m\geq 3\\ \text{or } n=m+2=3,5 \text{ or } 9.\end{array} \tag{21}$$

(One needs S^{n-2} to be parallelizable to be formally immersible in $\mathbb{R}^{n-2}$, which is only true for S^1, S^3, and S^7.) When the codimension $n-m=2$, the hairy graph-complexes are no more of finite type and their elements are infinite series of graphs. The graph-complexes in question are considered as completed pronilpotent L_∞-algebras, the completion being taken with respect to the complexity filtration; see the proof of Lemma 3.7. Since the L_∞-structure of $\mathrm{HGC}_{\bar{A}_m,n}$ is abelian, each space (20) is a product of Eilenberg–MacLane spaces

$$T_\infty\overline{\operatorname{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)^{\mathbb{Q}}\simeq\prod_{i=0}^{\infty}K\big(H_i(\mathrm{HGC}_{\bar{A}_m,n}),i\big),\quad n-m\geq 2.$$

In particular, this means that, for $n-m\geq 2$,

$$\pi_0 T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^m,\mathbb{R}^n)^{\mathbb{Q}} = \mathsf{MC}(\mathrm{HGC}_{\bar{A}_m,n})/{\sim} = H_0(\mathrm{HGC}_{\bar{A}_m,n}).$$

The statements of Theorem 3.4 and Corollary 3.6 hold for any m and n; in particular, they are also true in codimension $n-m=2$. The inclusion $U^t\oplus \mathrm{HGC}_{\bar{A}_m,n}\subset \mathrm{HGC}_{A_m,n}$ is a quasi-isomorphism of filtered (by complexity) completed L_∞-algebras, which induces a quasi-isomorphism of associated graded complexes. By the generalized Goldman–Millson theorem [10], this inclusion induces an equivalence of simplicial sets

$$\mathsf{MC}_\bullet(U^t\oplus \mathrm{HGC}_{\bar{A}_m,n})\simeq \mathsf{MC}_\bullet(\mathrm{HGC}_{A_m,n}).$$

As a consequence, in the range of equivalence (21), one has

$$\begin{aligned}\pi_0 T_\infty \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)^{\mathbb{Q}} &= \mathsf{MC}(\mathrm{HGC}_{A_m,n})/{\sim} = \mathsf{MC}(U^t\oplus \mathrm{HGC}_{\bar{A}_m,n})/{\sim}\\ &= \mathsf{MC}(\mathrm{HGC}_{\bar{A}_m,n})/{\sim} = H_0(\mathrm{HGC}_{\bar{A}_m,n}) = H_0(\mathrm{HGC}_{A_m,n}).\end{aligned}$$

Indeed, for $n-m\geq 3$, the third and last equalities are true by degree reasons (see [13, Cor. 5.2]), while for $n-m=2$ and n odd, U^t is a direct summand one-dimensional L_∞-subalgebra of $U^t\oplus \mathrm{HGC}_{\bar{A}_m,n}$ (spanned by L of degree 1). Thus, one has

$$T_\infty \overline{\mathrm{Emb}}(S^{n-2},\mathbb{R}^n)^{\mathbb{Q}} \simeq K(\mathbb{Q},1)\times T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-2},\mathbb{R}^n)^{\mathbb{Q}},\quad n=3,\,5 \text{ or } 9.$$

One does not know yet how to express algebraically $T_\infty \overline{\mathrm{Emb}}(S^m,\mathbb{R}^n)^{\mathbb{Q}}$ beyond the range of (21). On the other hand, the equivalence (20) had been proved earlier by B. Fresse and the authors in [12, Thm. 1]. In [12, Cor. 5 and Cor. 8], we similarly expressed $T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)^{\mathbb{Q}}$ and $T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n},\mathbb{R}^n)^{\mathbb{Q}}$ using the same, up to a degree one shift, graph-complex GC_n, the usual Kontsevich graph-complex of bald (no hairs) graphs endowed in both cases with the abelian L_∞-structure. The reason we get a smaller complex for $n-m=1$ is the relative non-formality of the little discs operads in codimension one [36]. As a consequence, we obtain [12, Eqn. (14)]

$$T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n},\mathbb{R}^n)^{\mathbb{Q}} \simeq \Omega T_\infty \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)^{\mathbb{Q}}.$$

Note that one also has

$$\mathrm{Diff}_\partial(D^n)\simeq \Omega\, \mathrm{Emb}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n)$$

(see [9, App., Sec. 5, Prop. 5], [5, Prop. 5.3]), which implies

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n},\mathbb{R}^n) := \mathrm{hofiber}(\mathrm{Diff}_\partial(D^n)\to \Omega^n\mathrm{SO}(n)) \simeq \Omega\, \overline{\mathrm{Emb}}_\partial(\mathbb{R}^{n-1},\mathbb{R}^n).$$

This means that, even though the Goodwillie–Weiss calculus is classically known to be applicable only in codimensions at least 3, it can still detect at least rationally the codimension one versus codimension zero rigidity of embeddings. In fact, very recently, different people started to question if the embedding calculus always fails in codimensions at most 2; as an interesting example, see [24].

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